Math 121 3.6 - 1st Implicit Differentiation

3.6-2nd Related Rates is an application of implicit diff.

Objective

1) Recognize when implicit differentiation can/should/must be done

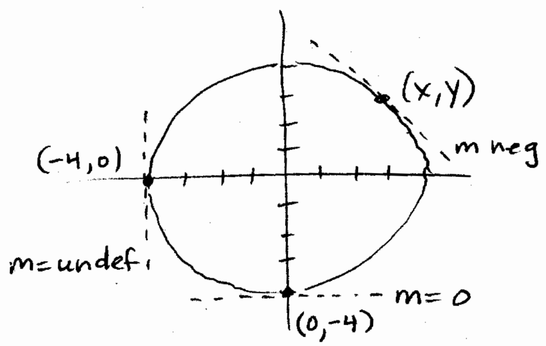
2) Find a derivative using the implicit differentiation process.

Implicit Differentiation is used when we want a derivative of an equation having x and y, rather than a function f(x).

- sometimes it is possible to isolate y and find the derivative using explicit methods from chapter 2.
- sometimes it is not possible to isolate y, and we must differentiate implicitly.

① Consider $x^2+y^2=16$.

Circle centered at (0,0) with radius $\sqrt{16}=4$



- If we do implicit differentiation, the derivative $\frac{dy}{dx}$ will be a function of both x and y, so that we can subst x and y to find the slope of the tangent line at that point.
- If we change the value of x, the value of y must change, too. So there is a rate of change of y with respect to x $= \frac{dy}{dx}$.

KEY POINT Because y changes with x, any time we take a derivative of y, we will use the chain rule and multiply that derivative by $\frac{dy}{dx}$.

e.x. $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$

Steps for finding a derivative implicitly:

1) Differentiate all terms of both sides of the equation, using the rules from chapter 2, and multiplying any derivative of y by $\frac{dy}{dx}$.

2) Use algebra to isolate $\frac{dy}{dx}$.

- collect all terms containing $\frac{dy}{dx}$ on one side of the =.
- collect remaining terms on the other side of the =.
- factor out $\frac{dy}{dx}$, if necessary.
- divide both sides to isolate $\frac{dy}{dx}$.

1) a) Find $\frac{dy}{dx}$ for $x^2+y^2=16$.

b) Find $\left.\frac{dy}{dx}\right|_{(-4,0)}$

c) Find $\left.\frac{dy}{dx}\right|_{(0,-4)}$

d) Find $\left.\frac{dy}{dx}\right|_{(2,2\sqrt{3})}$

a) Differentiate with respect to x:

$$\underbrace{2x}_{\text{power rule on } x^2} + \underbrace{2y}_{\text{power rule on } y^2}\cdot\underbrace{\frac{dy}{dx}}_{\text{chain rule on derivative of } y} = \underbrace{0}_{\text{constant rule}}$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$$ divide both sides by $2y$ to isolate $\frac{dy}{dx}$

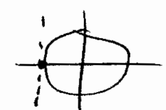
$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$ simplify

b) $$\left.\frac{dy}{dx}\right|_{(-4,0)} = \left.\frac{-x}{y}\right|_{(-4,0)}$$

$$= \frac{-(-4)}{0}$$

$$= \frac{4}{0}$$

$$= \boxed{\text{undefined}}$$ vertical tangent.

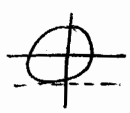


yes.

c) $$\left.\frac{dy}{dx}\right|_{(0,-4)} = \left.\frac{-x}{y}\right|_{(0,-4)}$$

$$= \frac{-0}{-4}$$

$$= \frac{0}{4} = \boxed{0}$$ horizontal tangent



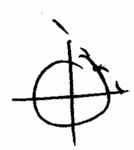
yes

d) $$\left.\frac{dy}{dx}\right|_{(2,2\sqrt{3})} = \left.\frac{-x}{y}\right|_{(2,2\sqrt{3})}$$

$$= \frac{-2}{2\sqrt{3}}$$

$$= \frac{-1}{\sqrt{3}}$$

$$= \boxed{\frac{-\sqrt{3}}{3}}$$



neg

yes.

Find derivatives using implicit differentiation.

(2) $x = y^3$

$1 = 3y^2 \cdot \frac{dy}{dx}$

$\boxed{\frac{1}{3y^2} = \frac{dy}{dx}}$

(3) $x^3y^5 + x = y$

product rule!

$x^3 \cdot \frac{d}{dx}(y^5) + \frac{d}{dx}(x^3) \cdot y^5 + 1 = 1 \cdot \frac{dy}{dx}$

$x^3 \cdot 5y^4 \frac{dy}{dx} + 3x^2y^5 + 1 = \frac{dy}{dx}$

$5x^3y^4 \frac{dy}{dx} - \frac{dy}{dx} = -3x^2y^5 - 1$

$\frac{dy}{dx}(5x^3y^4 - 1) = -3x^2y^5 - 1$

$\boxed{\frac{dy}{dx} = \frac{-3x^2y^5 - 1}{5x^3y^4 - 1}}$

(4) Find derivative implicitly.

$y^4 + x^4 - 2x^2y^2 = 9.$

b) Evaluate it at $x = 2$, $y = 1$.

$4y^3 \frac{dy}{dx} + 4x^3 - 2\left[x^2 \cdot \frac{d}{dx}(y^2) + 2x \cdot y^2\right] = 0$

$4y^3 \frac{dy}{dx} + 4x^3 - 2\left(x^2 \cdot 2y \frac{dy}{dx} + 2xy^2\right) = 0$

$4y^3 \frac{dy}{dx} + 4x^3 - 4x^2y \frac{dy}{dx} - 4xy^2 = 0$

$$4y^3 \frac{dy}{dx} - 4x^2y \frac{dy}{dx} = 4xy^2 - 4x^3$$

$$\frac{dy}{dx}\left(4y^3 - 4x^2y\right) = 4xy^2 - 4x^3$$

$$\frac{dy}{dx} = \frac{4xy^2 - 4x^3}{4y^3 - 4x^2y}$$

$$= \frac{4x(y^2 - x^2)}{4y(y^2 - x^2)}$$ factor and cancel

$$= \boxed{\frac{x}{y}}$$

b) $\left.\frac{x}{y}\right|_{(2,1)}$

$= \frac{2}{1}$

$= \boxed{2}$

④ c) slope of tangent line at $(2,1) = \left.\frac{dy}{dx}\right|_{(2,1)} = 2$ (from part b)

$m = 2$

point $(2,1)$ given

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$\boxed{y = 2x - 1}$

(5) A <u>demand function</u> $x(p)$ tells how many of a product (x) will be required if offered for sale at price p.

a) For the demand equation $x = \sqrt{1900-p^3}$, find $\frac{dp}{dx}$.

b) Evaluate $\left.\frac{dp}{dx}\right|_{p=10}$ and interpret result.

$x = \sqrt{1900-p^3}$

We can avoid the chain rule if we differentiate implicitly.

$x^2 = 1900 - p^3$

square both sides

$2x = 0 - 3p^2 \frac{dp}{dx}$

differentiate.

$\boxed{-\frac{2x}{3p^2} = \frac{dp}{dx}}$

isolate $\frac{dp}{dx}$

b) Notice: They provided only $p=10$. We need x-value!
Substitute $p=10$ into the original equation.

$$x = \sqrt{1900 - p^3}$$
$$x = \sqrt{1900 - 10^3}$$
$$x = \sqrt{900}$$
$$x = 30.$$

Evaluate $\left.\dfrac{-2x}{3p^2}\right|_{(x=30,\, p=10)}$

$$= \frac{-2(30)}{3(10)^2}$$
$$= \frac{-60}{300}$$
$$= \frac{-1}{5}$$
$$= -.2$$

When $p=10$ (The price is \$10), demand is 30 (items sold).
= Interpretation of original equation.

When price is \$10, $\frac{dp}{dx} = -0.2$.

units of $\frac{dp}{dx}$ = $\dfrac{\text{units of price } p}{\text{units of } x}$ = \$ per item sold.

So $-\$0.2/\text{item}$. means ....

$$\boxed{\text{When the price is \$10, decreasing the price \$0.20 results in one more item sold.}}$$